Behavior of the Escape Rate Function in Hyperbolic Dynamical Systems

Mark Demers[*] Paul Wright[†]

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Abstract

For a fixed initial reference measure, we study the dependence of the escape rate on the hole for a smooth or piecewise smooth hyperbolic map. First, we prove the existence and Hölder continuity of the escape rate for systems with small holes admitting Young towers. Then we consider general holes for Anosov diffeomorphisms, without size or Markovian restrictions. We prove bounds on the upper and lower escape rates using the notion of pressure on the survivor set and show that a variational principle holds under generic conditions. However, we also show that the escape rate function forms a devil's staircase with jumps along sequences of regular holes and present examples to elucidate some of the difficulties involved in formulating a general theory.

Consider a map $f : M \circlearrowleft$ of a measure space M in which a set $H \subset M$ is identified as a *hole*. We keep track of a point's orbit until it enters H; once this happens, it disappears and is not allowed to return.

One of the first quantities of interest for such open systems is the rate of escape of mass from the system after it is initially distributed according to a fixed reference measure, such as Lebesgue measure. More precisely, given a measure μ on M, the (exponential) *escape rate* of μ is $-\rho$, where

$$\rho = \rho(H, \mu) = \lim_{n\to\infty} \frac{1}{n} \log \mu(\cap_{i=0}^{n} f^{-i}(M\setminus H)), \tag{1}$$

if this limit exists. We write $\underline{\rho}$ and $\overline{\rho}$ for the lim inf and lim sup of the right hand side of (1), respectively, so that $-\underline{\rho}$ is the upper escape rate, and $-\overline{\rho}$ is the lower escape rate.

Note that $\rho \le 0$, so that the escape rate is non-negative. Also, although we usually suppress the parameters, ρ depends on both the hole, H, and on the measure, μ. In the present work, μ will always be either Lebesgue measure or the Sinai, Ruelle, Bowen (SRB) measure for f, and we study the behavior of the escape rate as a function of the hole.

Many works on systems with holes have focused on the existence of quasi-invariant measures with physical properties by assuming either the existence of a finite Markov partition

[*]Department of Mathematics and Computer Science, Fairfield University, Fairfield, USA. Email: mdemers@fairfield.edu. This research is partially supported by NSF grant DMS-0801139.

[†]Department of Mathematics, University of Maryland, College Park, USA. Email: paulrite@math.umd.edu.

[PY, Ce, CMS1, CM1] or that the holes are quite small [CMT1, LiM, CV, BDM, DWY1]. The derivative of the escape rate in the zero-hole limit has also received attention recently [BY, KL2, FP].

In this paper we consider *both* small and large holes and make no Markovian assumptions on the dynamics of the open systems. Let H_t be a 1-parameter family of holes varying continuously with t. Let $\rho(t) = \rho(H_t, \mu)$, when defined. In this context, we focus on two related questions.

1. **Is $t \mapsto \rho(t)$ continuous?** If so, does it possess a higher degree of regularity?[1]

2. **What is the overall structure of the escape rate function?** Are there qualitative differences in the escape rate function as we move out of the small hole regime?

The current understanding of the large hole regime is poor compared to the understanding of the small hole regime. This is because, for small holes, we may consider the open system as a perturbation of the closed system, where no mass escapes. In particular, perturbative spectral arguments have proven useful in studying this regime for certain systems, see for example [LiM, DL, KL1, DWY1]. In this paper, we extend such developments to prove the existence and Hölder continuity of the escape rate for systems with small holes admitting Young towers. Young towers are often used to study hyperbolic systems where a clear spectral picture is unavailable for the original map but is available for a tower extension. In order to prove our results, we construct a perturbative framework in the tower setting and then show that the results proved for the tower map can be pushed back to the original system. For this, we use the norms for the hyperbolic transfer operator on the tower introduced in [D2].

A second motivation of the present work is to make further headway in the study of the large hole regime, without making overly restrictive assumptions on the hole, such as that it be an element of a finite Markov partition for f. In order to do this, we obtain bounds on the escape rate using a variational principle. Such an approach for studying escape rates in uniformly hyperbolic systems was first introduced by Bowen [Bo2] and Young [Y1] and was recently developed for systems lacking uniform hyperbolicity [DWY2].

Since the general situation for systems with larger holes is unclear, we restrict our attention here to Anosov diffeomorphisms. We prove general bounds on the upper and lower escape rates using the notion of pressure on the survivor set and show that a variational principle holds under generic conditions; however, we also show that the escape rate function can form a devil's staircase, with jump discontinuities, for sequences of regular holes and present examples to elucidate some of the difficulties involved in formulating a general theory.

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[1] If the escape rate is not defined on an open interval of t, we can ask about the continuity of $\underline{\rho}$ and $\overline{\rho}$.

1 Statement of Results

All of the results in this work concern a map $f : M \circlearrowleft$, a C^2 or piecewise C^2 map of a Riemannian manifold. Throughout, μ denotes Lebesgue measure on M, which we assume is finite[2].

We consider holes $H \subset M$ that are open. The set of all such holes in M is denoted by $\mathcal{H}$. With $H \in \mathcal{H}$ fixed, we define $M^n = \cap_{i=0}^n f^{-i}(M \backslash H)$ and $\mathring{f}^n = f^n|_{M^n}$, for $n \geq 1$, to represent the dynamics of f restricted to the set of points that have not escaped by time n. Note that once H is introduced, its boundary ∂H must give rise to singularities for $\mathring{f}$, even if f is smooth. To this end, if $\mathcal{S}$ denotes the singularity set for f (which could be empty), we consider the enlarged singularity set given by $\mathcal{S}_H = \mathcal{S} \cup \partial H$.

A. Systems Admitting Young Towers

Our first result addresses the regularity of the escape rate for small holes when the system admits a Young tower as introduced in [Y2]. We define these towers in detail in Section 3, but we will introduce them now, as certain technical aspects of these towers are necessary for precisely stating our result.

In brief, the tower map $F : \Delta \circlearrowleft$ is a type of countable Markov extension of $f : M \circlearrowleft$ built from a return time function R defined on a hyperbolic set $\Lambda \subset M$. Returns are only defined when the returning set has a hyperbolic Markov structure.

The decay rate of the quantity $\mu(x \in \Lambda : R(x) > n)$ is crucial to determining the statistical properties of f. All of the towers we consider have *exponential tails*, i.e. there exist constants $C > 0$, $\theta < 1$ such that $\mu(R > n) \leq C\theta^n$ for all $n \geq 0$. In this case, under certain mixing assumptions on f, it may be possible to construct the tower so that a spectral gap exists for the transfer operator $\mathcal{L}_F$ associated with F acting on a certain space of distributions. By this, we mean that 1 is a simple eigenvalue for $\mathcal{L}_F$, and the rest of the spectrum is contained in a disk of radius $r < 1$. In this situation, an SRB measure ν_{SRB} for f can be constructed on M and many strong statistical properties for (f, ν_{SRB}) can be derived (see [Y2, D2]).

Let $\pi : \Delta \to M$ denote the canonical projection satisfying $f \circ \pi = \pi \circ F$. We say a tower (F, Δ) *respects the hole* H if the following conditions are satisfied:

(H.1) $\pi^{-1}H$ is the union of countably many elements in the Markov partition for F.

(H.2) The set Λ from which the base of the tower is constructed consists of points $x \in M \setminus H$ that approach $\mathcal{S}_H$ slower than a fixed exponential rate, i.e. there are constants $\delta > 0$, $\xi_1 > 1$, such that for all $n \geq 0$ and all $x \in \Lambda$, $d(f^n x, \mathcal{S}_H) \geq \delta \xi_1^{-n}$.

The notion of a tower respecting a hole has been used in a variety of settings starting with [D1]; for hyperbolic systems, the condition **(H.2)** was introduced in [DWY1]. In applications, **(H.1)** is ensured in part by adjoining the boundary of the hole to the singularity set, resulting in $\mathcal{S}_H$. Because this enlarged singularity set can cause unbounded changes in the return times to Λ, towers that respect holes must be constructed separately, even when towers have previously been constructed for the system without consideration of the hole.

[2] When f is only piecewise C^2, it is sometimes convenient to consider M to be a countable union of disjoint components, and so our assumptions do not exclude this possibility.

Once the hole has been introduced, we can ask if the spectral gap of $\mathcal{L}_F$ persists for the transfer operator $\mathcal{L}_{\mathring{F}}$ corresponding to the tower map with hole, $\mathring{F}$. By the persistence of the spectral gap for $\mathcal{L}_{\mathring{F}}$, we mean that $\mathcal{L}_{\mathring{F}}$ has a simple real eigenvalue $\mathfrak{r} \le 1$, and the rest of the spectrum of $\mathcal{L}_{\mathring{F}}$ is still contained in a disk of radius r, with $r < \mathfrak{r}$.

The existence of such a spectral gap implies that the escape rate equals $-\log \mathfrak{r}$ for a large class of initial measures on M. In addition, there is a unique physical quasi-invariant probability measure μ_H that is the analog of the physical SRB measure for the system f without holes. This quasi-invariant measure satisfies $\mathring{f}_* \mu_H = \mathfrak{r} \mu_H$, and there is a large class of initial measures on M that limit on μ_H if they are pushed forward under the action of $\mathring{f}$ and then renormalized back into probability measures (see [DY] for a survey of these techniques).

We call H_0 an *infinitesimal hole* if $\mu(H_0) = 0$ and H_0 can be realized as a limit of holes $H \in \mathcal{H}$ in the topology induced by the Hausdorff metric.

Let $\{H_t\}_{t \in I} \subset \mathcal{H}$, where I is some interval, be a family of holes. We call the family *well-parametrized* if

(i) $0 \in I$ and H_0 is an infinitesimal hole;

(ii) $\mathrm{dist}(\partial H_t, \partial H_{t'}) \le |t - t'|$, where distance is measured in the Hausdorff metric.

Suppose that f has a unique invariant SRB measure ν_{SRB}. In Section 3.5 we prove the following theorem.

Theorem 1. (Continuity of the Escape Rate for Small Holes) *Let $\{H_t\}_{t \in I}$ be a well-parametrized sequence of holes in $\mathcal{H}$. Suppose that for each $t \in I$, (f, M) admits a tower respecting H_t. If for some $\varepsilon > 0$, the towers satisfy the uniformity conditions* **(U)** *of Section 3.5 for all $t \in [0, \varepsilon]$, then there exists $0 < \delta \le \varepsilon$ such that $\rho(t) = \rho(H_t, \nu_{SRB})$ exists for each $t \in [0, \delta]$, and $t \mapsto \rho(t)$ is a Hölder continuous function. In addition, for $t \in [0, \delta]$ a unique physical quasi-invariant measure μ_{H_t} exists, and these measures vary Hölder continuously with t.*

The measures varying Hölder continuously means that there exist $C, \alpha > 0$ such that,

$$|\mu_{H_t}(\varphi) - \mu_{H_{t'}}(\varphi)| \le C |t - t'|^\alpha |\varphi|_{C^0(M)} \quad \text{for all } \varphi \in \mathcal{C}^0(M).$$

The basic idea of our proof is as follows: Given H_t and $H_{t'}$ with $|t - t'|$ small, we construct a tower over $f : M \circlearrowleft$ that respects *both* H_t and $H_{t'}$. On the tower, the lifted holes $\pi^{-1} H_t$ and $\pi^{-1} H_{t'}$ are close together, and so the spectral picture changes little when we consider one open transfer operator versus the other. To make this precise, we use the perturbative framework of [KL1] and the hyperbolic norms introduced in [D2]. Our use of these hyperbolic norms on Δ simplifies the argument greatly and allows us to avoid working with a second induced object, the quotient tower $\overline{\Delta}$ (see Remark 3.2).

The analogue of Theorem 1 has been shown to hold in [DL] for Anosov diffeomorphisms and some piecewise hyperbolic systems with bounded derivative in two dimensions using a different approach.

An application to the 2D periodic Lorentz gas. Let $f : M \circlearrowleft$ be the billiard map associated with a two-dimensional periodic Lorentz gas with finite horizon whose scatterers are bounded by $\mathcal{C}^3$ curves with strictly positive curvature. In [DWY1], holes are introduced into M that are derived from two types of holes in the billiard table X: An open segment on

the boundary of one of the scatterers or an open convex set in X whose closure is disjoint from any of the scatterers. If $\sigma \subset X$ is one such hole, it induces a hole $H_\sigma \subset M$ which is labeled as a hole of Type I or Type II respectively. For a detailed description of the geometry of these holes in M, see [DWY1, Section 3.1]. The Young towers for this class of maps constructed in [DWY1] satisfy the assumptions of Theorem 1, yielding the following corollary.

Corollary 1.1. *Suppose f is the billiard map described above and let $\{H_t\}_{t\in I}$ be a well-parametrized sequence of holes of Type I or Type II. For t sufficiently small, both the escape rate $\rho(t)$ and the physical quasi-invariant measures μ_{H_t} vary Hölder continuously with t.*

B. Large and Small Holes for Anosov Diffeomorphisms

Our next result concerns the general behavior of the escape rate as a function of both small and large holes. Because the general picture for large holes is unclear, we limit our attention to Anosov diffeomorphisms. Since we cannot rely on perturbative spectral arguments, we use a variational principle to obtain bounds on the escape rate. Such an approach for investigating escape rates was first used in [Bo2]; see [DWY2] for a summary of further developments in this direction.

Let M be a compact Riemannian manifold and let $f : M \circlearrowleft$ be a $C^{1+\epsilon}$ Anosov diffeomorphism. Note that, unlike in Section **A** above, for any $H \in \mathcal{H}$, $\mathcal{S}_H = \partial H$ is now automatically compact, since M is. As before, μ denotes Lebesgue measure on M. The *survivor set*, $\Omega = \Omega(H) = \cap_{n\in\mathbb{Z}} f^n(M \setminus H)$, is the set of points whose forward and backward orbits never enter H. We do not place any *a priori* assumptions on the mixing properties of f.

Given an f-invariant measure on M, we define

$$P_\nu = h_\nu(f) - \int \chi^+ \, d\nu,$$

where $h_\nu(f)$ is the metric entropy of ν, and χ^+ is the sum of positive Lyapunov exponents of f, counted with multiplicity. In [CM1, CM2], the authors prove that for holes that are elements of a finite Markov partition, an *escape rate formula* holds, i.e. there exists an ergodic, f-invariant measure ν such that $\rho(\mu) = P_\nu$. These results were extended to small, non-Markov holes by approximation in [CMT2].

Note that any $\mathring{f}$-invariant measure must have its support contained inside Ω, and so we define $\mathcal{I}(\Omega)$ to be the set of ergodic, f-invariant measures whose support is contained in Ω.

Given a class of f-invariant measures $\mathcal{C}$ on Ω, we define the *pressure* on Ω to be $\mathcal{P}_\mathcal{C} = \sup_{\nu\in\mathcal{C}} P_\nu$. We say that f satisfies a *full variational principle* if $\rho(\mu) = \mathcal{P}_\mathcal{C}$, for an appropriate class of measures $\mathcal{C}$. Ideally, the class should be as large as possible. Let $N_\varepsilon(A)$ denote the ε-neighborhood of a set A. Following [DWY2, Section 2.I], we define

$$\begin{aligned}\mathcal{G}(\Omega) = \{\nu \in \mathcal{I}(\Omega) : &\text{ The following holds for } \nu\text{-a.e. } x\text{: given } \gamma > 0,\ \exists r = r(x,\gamma) > 0 \\ &\text{such that } N_{re^{-\gamma i}}(f^i x) \subset M \setminus H \text{ for all } i \geq 0\}.\end{aligned}$$

A standard Borel-Cantelli argument shows that if $\nu \in \mathcal{I}(\Omega)$ has the property that for some $C, \alpha > 0$, $\nu(N_\varepsilon(\partial H)) \leq C\varepsilon^\alpha$ for all $\varepsilon > 0$, then $\nu \in \mathcal{G}(\Omega)$.

A full variational principle and an escape rate formula are proved for hyperbolic systems admitting Young towers respecting small holes in [DWY2], with $\mathcal{G}(\Omega)$ as the class of measures over which the supremum is taken. In this section, we take a more general approach and study the relationship between escape rate and pressure without placing restrictions on the size or placement of the hole. All results stated in this section are proved in Section 2.

Theorem 2. *Let $f : M \circlearrowleft$ be a $C^{1+\varepsilon}$ Anosov diffeomorphism with hole $H \in \mathcal{H}$. Then*

$$\mathcal{P}_{\mathcal{G}} \ \leq \ \underline{\rho}(H,\mu) \ \leq \ \overline{\rho}(H,\mu) \ \leq \ \mathcal{P}_{\mathcal{I}} \ .$$

If in addition $\partial H \cap \Omega(H) = \emptyset$, then $\rho(H,\mu)$ is well-defined and equals $\mathcal{P}_{\mathcal{G}} = \mathcal{P}_{\mathcal{I}}$.

Note that since $\Omega(H)$ and ∂H are both compact, the condition $\partial H \cap \Omega(H) = \emptyset$ is equivalent to $d(\partial H, \Omega(H)) > 0$, where d is distance in the Hausdorff metric. Hence, in this case, $\mathcal{G} = \mathcal{I}$.

In what follows, we consider holes in $\mathcal{H}$ that have a finite number of connected components. Let $\mathcal{H}_{\text{fin}}$ denote this collection; $\mathcal{H}_{\text{fin}}$ inherits the topology induced by the Hausdorff metric. In the two-dimensional case, we call a hole $H \in \mathcal{H}_{\text{fin}}$ *regular* if its boundary is comprised of a finite number of local stable and unstable manifolds.

Theorem 2 points to the existence of invariant measures for $\mathring{f}$ that give too much weight to small neighborhoods of ∂H as a source of potential problems. Our next set of results illustrates two points. (1) The condition $\partial H \cap \Omega(H) = \emptyset$ is quite general: It holds for an open (and in dimension two, dense) set of holes and for a full measure set of parameters along sequences of regular holes; (2) exceptional situations do occur. Indeed, these exceptions cause the escape rate to vary - otherwise, it remains locally constant. Proposition 1.7 presents situations where the inequalities in Theorem 2 are strict. Although the examples are contrived, they need to be taken into account in the formulation of general results.

Proposition 1.2. *Let $f : M \circlearrowleft$ be as in Theorem 2. Then*

(a) the set of $H \in \mathcal{H}_{\mathit{fin}}$ that satisfy $\partial H \cap \Omega(H) = \emptyset$ is open in $\mathcal{H}_{\mathit{fin}}$;

(b) if $\dim(M) = 2$ and f is topologically transitive, then the set of $H \in \mathcal{H}_{\mathit{fin}}$ that satisfy $\partial H \cap \Omega(H) = \emptyset$ is dense in $\mathcal{H}_{\mathit{fin}}$.

Remark 1.3. *In the setting of Proposition 1.2(b), for each $H \in \mathcal{H}_{\mathit{fin}}$ and $\varepsilon > 0$, in fact there exists a regular hole H_ε such that $d(H, H_\varepsilon) < \varepsilon$ and $d(\partial H_\varepsilon, \Omega(H_\varepsilon)) > 0$. This is shown in our proof.*

Proposition 1.4. *Let $\dim(M) = 2$, f be topologically transitive and suppose $\{H_t\}_{t \in I} \subset \mathcal{H}_{\mathit{fin}}$ is a sequence of regular holes with a fixed number of connected components satisfying:*

(i) the number of smooth components of ∂H_t is uniformly bounded on I;

(ii) $t \mapsto \partial H_t$ is continuous;

(iii) For any subinterval $J \subseteq I$ and any curve γ locally transverse to $\{\partial H_t\}_{t \in J}$, if $E \subset J$ has positive Lebesgue measure in I, then $\{\gamma \cap \partial H_t\}_{t \in E}$ has positive Lebesgue measure on γ.

Then $\Omega(H_t) \cap \partial H_t = \emptyset$ for an open and dense set of $t \in I$ and the exceptional set has zero Lebesgue measure in I.

Note that the sequence $\{H_t\}_{t\in I}$ is neither assumed to converge to a point nor to be monotonic.

Corollary 1.5. *Let $f : M \circlearrowleft$ and $\{H_t\}_{t\in I}$ be as in Proposition 1.4 and let $-\underline{\rho}(t)$ and $-\overline{\rho}(t)$ denote the upper and lower escape rates from $M \setminus H_t$ with respect to Lebesgue. Then*

(a) $\rho(t)$ exists and is locally constant on an open and full measure set of t.

Now assume that $\{H_t\}_{t\in I}$ is monotonically increasing. Then

(b) the functions $t \mapsto \underline{\rho}(t)$ and $\overline{\rho}(t)$ are monotonically decreasing and each forms a devil's staircase, possibly with jumps;

(c) $\underline{\rho}(\cdot)$ and $\overline{\rho}(\cdot)$ are in general neither upper nor lower semi-continuous once they are out of the small hole regime;

(d) if $\overline{\rho}(\cdot)$ is lower semi-continuous at t, then $\rho(t)$ exists;

(e) if $\underline{\rho}(\cdot)$ is upper semi-continuous at t, then $\rho(t)$ exists.

Taken together, statements (d) and(e) above imply that $\rho(t)$ typically exists even when $\partial H_t \cap \Omega(H_t) \neq \emptyset$. The only values of t at which $\rho(t)$ may not exist are those at which $\overline{\rho}(t)$ and $\underline{\rho}(t)$ jump and fail to be lower and upper semi-continuous, respectively. This can occur at most countably many times along the sequence.

Remark 1.6. *If one considers the recent results in [BY, KL2, FP] regarding the existence of the derivative of $\rho(t)$ in the zero hole limit in a number of hyperbolic settings, the picture of $\rho(t)$ that emerges from Corollary 1.5 is rather surprising. It indicates that along sequences of regular holes, $\rho(t)$ cannot be smooth on any interval containing 0: Indeed $\rho(t)$ cannot even be absolutely continuous on any interval on which it is not constant.*

Proposition 1.7. *There are examples of Anosov diffeomorphisms with regular holes where*

(a) $\mathcal{P}_\mathcal{G} < \rho(\mu) = \mathcal{P}_\mathcal{I}$; (b) $\mathcal{P}_\mathcal{G} = \rho(\mu) < \mathcal{P}_\mathcal{I}$; and (c) $\mathcal{P}_\mathcal{G} < \underline{\rho}(\mu) \leq \overline{\rho}(\mu) < \mathcal{P}_\mathcal{I}$.

Remark 1.8. *We conclude with an observation on large versus small holes in hyperbolic systems. Evidently, no invariant measure with pressure close enough to 0 can be too concentrated near ∂H for a large class of small holes since $t \mapsto \rho(t)$ is Hölder continuous in the setting of Theorem 1. On the other hand, for larger holes invariant measures which maximize pressure can live on ∂H and create jumps in the escape rate as demonstrated by Corollary 1.5 and Proposition 1.7.*

2 Proofs of Anosov Results

2.1 Proof of Theorem 2

Recall that the conditions $\partial H \cap \Omega(H) = \emptyset$ and $d(\partial H, \Omega(H)) > 0$ are equivalent since both sets are compact. In this case $\mathcal{G} = \mathcal{I}$ so that $\mathcal{P}_{\mathcal{G}} = \rho(H, \mu) = \mathcal{P}_{\mathcal{I}}$ once the inequalities in the statement of Theorem 1 are proved. That $\mathcal{P}_{\mathcal{G}} \leq \underline{\rho}(H, \mu)$ follows from [DWY2, Theorem A].

To finish the proof, it suffices to show that $\overline{\rho}(H, \mu) \leq \mathcal{P}_{\mathcal{I}}$. To this end, approximate H by a sequence of increasing holes $H_n \subset H$, $H_n \in \mathcal{H}$, such that each H_n is a union of finitely many elements of a Markov partition for f. Since f admits finite Markov partitions with arbitrarily small diameter, we can choose the sequence so that $\cup_{n=1}^{\infty} H_n = H$.

We set $\mathring{M}_n = M \setminus H_n$ and in general denote by the subscript n objects associated with H_n. Note that $\mathring{M}_n$ is a decreasing sequence of closed sets converging to $\mathring{M} = M \setminus H$. The same is true of Ω_n and Ω. Since H_n is a Markov hole, the escape rate $\rho_n = \rho(H_n, \mu)$ is well-defined and there exists $\nu_n \in \mathcal{I}(\Omega_n)$ such that $\rho_n = P_{\nu_n}$ [CM2].

Let ν be a limit point of the ν_n. Then ν is an f-invariant measure whose support is contained in Ω, but ν need not be ergodic. (Even if it were, it would not necessarally be an element of $\mathcal{G}$.) Since $x \mapsto \log|\det(Df|_{E^u})(x)|$ is a continuous function, we have $\lim_{n\to\infty} \int \chi^+ d\nu_n = \int \chi^+ d\nu$.

In addition, $h_\nu(f) \geq \limsup_{n\to\infty} h_{\nu_n}(f)$ due to the expansiveness of f. We include the brief proof here for convenience. Since f is expansive, there exists $\varepsilon > 0$ such that if ξ is a finite measurable partition of M with $\mathrm{diam}(\xi) < \varepsilon$, then $h_\eta(f, \xi) = h_\eta(f)$ for any invariant Borel measure η [Bo1]. Fix such a partition ξ with $\nu(\partial\xi) = 0$. Let $H_\eta(\xi_k)$ denote the entropy of the partition $\bigvee_{i=-k}^{k} f^i\xi$ with respect to a measure η, and for $\delta > 0$ choose k such that $\frac{1}{k} H_\nu(\xi_k) \leq h_\nu(f) + \delta$. Then since $\frac{1}{k} H_\eta(\xi_k)$ is a decreasing function of k for any η, we have

$$\limsup_{n\to\infty} h_{\nu_n}(f) = \limsup_{n\to\infty} h_{\nu_n}(f, \xi) \leq \limsup_{n\to\infty} \frac{1}{k} H_{\nu_n}(\xi_k) = \lim_{n\to\infty} \frac{1}{k} H_\nu(\xi_k) \leq h_\nu(f) + \delta,$$

which proves the claim, since $\delta > 0$ is arbitrary.

We have shown that

$$P_\nu \geq \limsup_{n\to\infty} P_{\nu_n} = \limsup_{n\to\infty} \rho_n \geq \overline{\rho},$$

where the last inequality is true by monotonicity: $\mathring{M}_n \supset \mathring{M}$ for each n. By the ergodic decomposition, there exists a measure π_ν on $\mathcal{I}(\Omega)$ such that $\nu = \int_{\mathcal{I}} \eta \, d\pi_\nu(\eta)$. In fact, since f and $\log|\det(Df|_{E^u})|$ are continuous, we have $h_\nu(f) - \int \chi^+ d\nu = \int_{\mathcal{I}} (h_\eta(f) - \int \chi^+ d\eta) \, d\pi_\nu(\eta)$ (see for example [W, Theorem 8.4]), so there must exist an ergodic measure $\eta \in \mathcal{I}(\Omega)$ such that $P_\eta \geq \overline{\rho}$. This finishes the proof of Theorem 2

2.2 Proof of Proposition 1.2

Throughout this section we assume that $H \in \mathcal{H}_{\mathrm{fin}}$ has only one connected component. The case of multiple connected components is handled similarly, one component at a time.

As a reminder, because Ω and ∂H are compact, the condition that $\partial H \cap \Omega(H) = \emptyset$ is equivalent to $d(\partial H, \Omega(H)) > 0$.

To prove statement (a), choose $H \in \mathcal{H}_{\text{fin}}$ such that $d(\partial H, \Omega(H)) > 0$. Let $B(x, \varepsilon)$ denote the ball of radius ε centered at x. For each $x \in \partial H$, let $\varepsilon(x)$ and $n(x)$ be such that $f^{n(x)}(B(x, \varepsilon(x))) \subset H$. Since ∂H is compact, we may choose $x_1, \ldots, x_k$ so that $U_0 := \cup_{i=1}^k B(x_i, \frac{1}{2}\varepsilon(x_i)) \supset \partial H$. Notice that $V = \cup_{i=1}^k f^{(n(x_i))}(B(x_i, \frac{1}{2}\varepsilon(x_i))) \subset H$ is a positive distance from ∂H. Let $U_1 \subseteq U_0$ be a neighborhood of ∂H that is a positive distance from $V \cup \Omega$.

Now consider a hole $H' \in \mathcal{H}_{\text{fin}}$ with survivor set Ω' such that $d(H, H') < \delta$ for some $\delta > 0$. By taking δ sufficiently small, we can ensure that $[(H' \setminus H) \cup (H \setminus H')] \subset U_1$. If $x \notin \Omega'$, then $f^j x \in H'$ for some j. Either $f^j x \in H$ or $f^j x \in U_0$, and in the latter case a further iterate of x eventually lies in $V \subset H \cap H'$. In both cases, $x \notin \Omega$, so $\Omega \subset \Omega'$. Similarly, $\Omega' \subset \Omega$. To conclude, because $\partial H' \subset U_1$ and U_1 is a positive distance from $\Omega = \Omega'$, we have $\partial H' \cap \Omega' = \emptyset$.

To prove statement (b), fix $H \in \mathcal{H}_{\text{fin}}$ and $\varepsilon > 0$ sufficiently small that H contains a ball of diameter at least 5ε. We approximate ∂H by a union of finitely many stable and unstable manifolds $\Gamma' = (\cup_i \gamma_i^s) \cup (\cup_i \gamma_i^u)$ such that $d(\Gamma', \partial H) < \varepsilon/2$ where $\gamma_i^{s(u)}$ denotes the ith (un)stable manifold. Let $B_\varepsilon \subset H$ be an open convex set such that $d(B_\varepsilon, H) > 2\varepsilon$.

Now fix γ_i^s and let $N_\varepsilon(\gamma_i^s)$ denote the ε neighborhood of γ_i^s in M. By transitivity of f, μ-almost every $x \in N_{\varepsilon/2}(\gamma_i^s)$ has a dense orbit. Choose such an x and let γ_x^s be the local stable manifold through x of length at least $2|\gamma_i^s|$. There exists an integer n_x and an open set $U_x \supset \gamma_x^s$ such that $f^{n_x}(U_x) \subset B_\varepsilon$.

Now we modify Γ' by replacing γ_i^s with γ_x^s and possibly lengthening or shortening the γ_j^u adjacent to γ_i^s so that the ends of γ_j^u which formerly ended on γ_i^s now end on γ_x^s. We trim the ends of γ_x^s as necessary. We continue this process with each γ_i^s in forward time and each γ_i^u in backward time. In this way, we construct Γ, a simple closed curve made up of finitely many stable and unstable manifolds γ_k, which enjoys the following properties: (i) $d(\Gamma, \partial H) < \varepsilon$; (ii) for each γ_k there exists an open set $U_k \supset \gamma_k$ and $n_k \in \mathbb{Z}$ such that $f^{n_k}(U_k) \subset B_\varepsilon$. Thus letting H_Γ denote the hole with boundary Γ and noting that $B_\varepsilon \subset H_\Gamma$ by construction, we have $d(\partial H_\Gamma, \Omega(H_\Gamma) > 0$ as required.

2.3 Proof of Proposition 1.4

Let $\{H_t\}_{t \in I}$ be a sequence of holes as described in the statement of Proposition 1.4. That the condition $\Omega(H_t) \cap \partial H_t = \emptyset$ holds on an open set in I follows from Proposition 1.2 since by assumption (ii) on the sequence, H_t varies continuously with t in the Hausdorff metric.

Thus it suffices to show that the exceptional set has Lebesgue measure 0 in I, for then an open set of full Lebesgue measure is necessarily dense. We do this by contradiction. Let $E = \{t \in I : \Omega(H_t) \cap \partial H_t \neq \emptyset\}$ and suppose $\ell(E) > 0$, where ℓ denotes Lebesgue measure on I.

We partition $\{\partial H_t\}_{t \in I}$ into finitely many boxes B_i sufficiently small that all the elements of $\{\partial H_t \cap B_i\}_{t \in I}$ are roughly parallel. This is possible due to assumption (i) and the fact that the stable and unstable foliations are Hölder continuous. Let $E_i = \{t \in I : \Omega(H_t) \cap \partial H_t \cap B_i \neq \emptyset\}$. At least one of these sets must satisfy $\ell(E_i) > 0$. Fix one such index and call it k. We suppose without loss of generality that the curves ∂H_t in B_k are all local stable manifolds.

Draw a curve γ in B_k that is uniformly transverse to each curve ∂H_t lying in B_k. We choose one point $x_t \in \Omega_t \cap \partial H_t \cap B_k$ for each $t \in E_k$. Due to property (iii) of $\{\partial H_t\}$, we

have $\mu_\gamma(\gamma \cap W^s_{\text{loc}}(x_t) : t \in E_k) > 0$, where μ_γ denotes arclength on γ.

Note that if the forward orbit of x is dense, then the entire stable manifold of x eventually falls into any open hole. Since $x_t \in \Omega(H_t)$, the curve $W^s_{\text{loc}}(x_t)$ cannot contain any forward dense points, for then x_t would fall into H_t under forward iteration as observed above. Integrating these curves over $t \in E_k$, we see that a positive μ-measure set of points in B_k do not have a dense forward orbit. This contradicts the fact that points with dense forward orbits have full measure in M. Thus $\ell(E) = 0$ as required.

The case in which the curves ∂H_t in B_k are local unstable manifolds is handled similarly using the fact that points with dense backward orbits have full measure as well.

2.4 Proof of Proposition 1.7

(a) Let $\dim(M) = 2$ and let p be a fixed point for f with expanding eigenvalue $\lambda > 1$, i.e., f is orientation preserving. Let $W^s_{\text{loc}}(p)$ denote the local stable manifold through p. Let U be a neighborhood of p, divided into two halves by $W^s_{\text{loc}}(p)$: the left half is in H and the right one not, so that $W^s_{\text{loc}}(p) \subset \partial H$. From this alone, we see that $\underline{\rho} \geq -\log \lambda$. We now construct f and H with two additional features: (i) Except for $W^s_{\text{loc}}(p)$, all points in the right half of U eventually fall into the hole; this can be arranged by having $W^u(p)$ run into H. (ii) By taking λ close enough to 1 and H large enough, we can arrange for $\mathcal{P}_{\mathcal{I}_p} < -\log \lambda$ where $\mathcal{I}_p \subset \mathcal{I}$ is the set of ergodic invariant measures supported on $\Omega \setminus \{p\}$. Thus $\mathcal{P}_{\mathcal{I}} = -\log \lambda$ and as noted earlier $\underline{\rho} \geq -\log \lambda$. But $\overline{\rho} \leq \mathcal{P}_{\mathcal{I}}$ by Theorem 2 so that ρ is well-defined and $\rho = \mathcal{P}_{\mathcal{I}} = -\log \lambda$. Also, since $\mathcal{G} \subset \mathcal{I}_p$, we have $\mathcal{P}_{\mathcal{G}} < \rho$.

(b) We use the same setup as in (a), but now $\lambda < -1$, i.e., f is orientation reversing. Now $p \in \Omega$ as before, but both halves of U on either side of $W^s_{\text{loc}}(p)$ fall into H in finitely many steps so that the escape rate is unrelated to the eigenvalue at p. We choose λ close enough to -1 that $\overline{\rho} < \mathcal{P}_{\mathcal{I}} = -\log|\lambda|$. We further require that: (i) H is a union of elements of a Markov partition for f; (ii) $(\partial H \setminus W^s_{\text{loc}}(p)) \cap \Omega = \emptyset$. By (i) and the results of [CM2], $\rho(\mu)$ is well-defined and there exists an ergodic invariant measure ν supported on Ω such that $P_\nu = \rho$. By (ii), $\nu \in \mathcal{G}$ so that by Theorem 2, $\mathcal{P}_{\mathcal{G}} = \rho$.

(c) We combine the two behaviors described in parts (a) and (b). Assume f is orientation preserving. Let p be a fixed point for f and let q, q' be an orbit of period two. Let μ_p denote the point mass at p and let ν_q denote the invariant measure supported on $\{q, q'\}$. Let V_q denote a neighborhood of q divided into two halves by $W^s_{\text{loc}}(q)$. Orienting stable manifolds in an approximately vertical direction, we label these two halves as V^ℓ_q and V^r_q for left and right. We define analogous objects for q'.

We choose a regular hole H with the following properties: (i) $W^s_{\text{loc}}(p) \subset \partial H$ and the neighborhood U of p is as described in (a); (ii) $W^s_{\text{loc}}(q) \cup W^s_{\text{loc}}(q') \subset \partial H$, $V^r_q \subset H$ and $V^\ell_{q'} \subset H$. Letting λ_p denote the expanding eigenvalue at p and using the same reasoning as in (a), we see from (i) that $\underline{\rho} \geq -\log \lambda_p$. We let $\mathcal{I}' = \mathcal{I} \setminus \{\mu_p, \nu_q\}$ and choose λ_p close enough to 1 such that $\mathcal{P}_{\mathcal{I}'} < -\log \lambda_p$ so that $\mathcal{P}_{\mathcal{G}} \leq \mathcal{P}_{\mathcal{I}'} < \underline{\rho}$.

Since f is orientation preserving, $f(V^\ell_q) \subset H$ and $f(V^r_{q'}) \subset H$ so that the full measure of $V_q \cup V_{q'}$ has escaped after one step. Thus $\overline{\rho}$ is independent of $\log \lambda_q$, the expanding Lyapunov exponent on the orbit q, q'. Taking λ_q close enough to 1, we can force $\overline{\rho} < -\log \lambda_q \leq \mathcal{P}_{\mathcal{I}}$.

2.5 Proof of Corollary 1.5

Proof of (a). It follows from Theorem 2 and Proposition 1.4 that $\rho(t)$ is locally constant on an open and full measure set of t since $\rho(t)$ exists and is locally constant around any t satisfying $d(\partial H_t, \Omega(H_t)) > 0$.

Proof of (b). Monotonicity of $\{H_t\}$ clearly implies monotonicity of both $\underline{\rho}_t$ amd $\overline{\rho}_t$ and their characterization as a devil's staircase follows since the derivative of each function exists and equals zero on an open subset of I of full measure by part (a).

We assume for the remainder of the proof that $\{H_t\}_{t\in I}$ forms an increasing sequence of holes so that $\underline{\rho}(t)$ and $\overline{\rho}(t)$ are (nonstrictly) decreasing functions of t. We prove (d) and (e) first and leave statement (c) for last.

Proof of (d). Assume $\overline{\rho}$ is lower semi-continuous at t_0. By part (b), there exists a sequence $t_n \downarrow t_0$ such that $\overline{\rho}(t_n) \uparrow \overline{\rho}(t_0)$, and by part (a), the t_n can be chosen so that $\rho(t_n)$ exists for each n, i.e. $\underline{\rho}(t_n) = \overline{\rho}(t_n)$. Thus

$$\overline{\rho}(t_0) = \lim_{n\to\infty} \overline{\rho}(t_n) = \lim_{n\to\infty} \underline{\rho}(t_n) \leq \underline{\rho}(t_0)$$

where in the last inequality we have used monotonicity of $\underline{\rho}$ since $t_n > t_0$ implies $H_{t_n} \supset H_{t_0}$ for each n. Since $\underline{\rho}(t_0) \leq \overline{\rho}(t_0)$ by definition, we have $\underline{\rho}(t_0) = \overline{\rho}(t_0)$ so that $\rho(t_0)$ exists.

Proof of (e). This is similar to part (d) except that we choose a sequence $t_k \uparrow t_0$ such that $\underline{\rho}(t_k) \downarrow \underline{\rho}(t_0)$ at a point where $\underline{\rho}$ is upper semi-continuous.

Proof of (c). Finally, we show that $\underline{\rho}(t)$ and $\overline{\rho}(t)$ can in fact have jumps at particular values of t. We refer to the examples constructed in the proof of Proposition 1.7. All notation is as in that proof.

Violation of lower semicontinuity. Let H_{t_0} be a hole satisfying the requirements of case (a) in the proof of Proposition 1.7. For $t < t_0$, take H_t to have the same boundary as H_{t_0} except for $W^s_{\text{loc}}(p)$. Here, we make the boundary of H_t be a local stable manifold running parallel to $W^s_{\text{loc}}(p)$ lying inside $U \cap H_{t_0}$ and varying continuously with t. Notice that since $\rho(t_0) = -\log \lambda$ in this example, shrinking H_t in this way does not change the escape rate for t close to t_0. Also, $\Omega(H_t) = \Omega(H_{t_0})$ since $U \cap \Omega(H_t) = \emptyset$. Thus $\rho(t) = -\log \lambda$ for all $t \in (t_0 - \varepsilon, t_0]$ for ε sufficiently small.

On the other hand, for $t > t_0$, we replace $W^s_{\text{loc}}(p)$ by a local stable manifold running parallel to $W^s_{\text{loc}}(p)$ lying in $U \setminus H_{t_0}$ so that p is no longer in $\Omega(H_t)$. Then $\mathcal{I}(H_t) = \mathcal{I}_p(H_{t_0})$ so that for some $\delta > 0$ and all $t > t_0$, $\mathcal{P}_{\mathcal{I}(H_t)} < -\log \lambda - \delta = \rho(t_0) - \delta$ by construction of p. By Theorem 2, we have $\underline{\rho}(t) \leq \overline{\rho}(t) \leq \mathcal{P}_{\mathcal{I}(H_t)} < \rho(t_0) - \delta$ for all $t > t_0$.

Violation of upper semicontinuity. Let H_t be the same as described in the previous step, except that H_{t_0} is a hole satisfying the requirements of case (b) in the proof of Proposition 1.7, i.e., the case when f is orientation reversing. Now the holes for $t > t_0$ satisfy $\rho(t) = \rho(t_0)$ since the full measure of U disappears in one step for all the H_t. For $t < t_0$, now a neighborhood of p survives for arbitrarily many steps so that λ dominates the escape, i.e., we choose λ so that $\overline{\rho}(t) \geq \underline{\rho}(t) \geq -\log|\lambda| > \rho(t_0) + \delta$ for some $\delta > 0$.

3 Proof of Theorem 1

We begin by reviewing some facts about Young towers from [Y2, D2]. We then use these facts to prove new results on the tower that we use to prove Theorem 1 in Section 3.5.

3.1 Generalized horseshoe respecting H

We recall the notion of a generalized horseshoe with infinitely many branches and variable return times. The existence of such a horseshoe leads immediately to the definition of a Young tower. We summarize here only the most important properties and refer the reader to [Y2, Section 1.1] for full details.

Following the notation in Section 1.1 of [Y2], we consider a smooth or piecewise smooth map $f : M \to M$, and let μ and μ_γ denote respectively the Riemannian measure on M and on γ where $\gamma \subset M$ is a submanifold. We say the pair (Λ, R) defines a *generalized horseshoe* if **(P1)**–**(P5)** below hold (see [Y2] for precise formulation):

(P1) Λ is a compact subset of M with a hyperbolic product structure, *i.e.*, $\Lambda = (\cup\Gamma^u)\cap(\cup\Gamma^s)$ where Γ^s and Γ^u are continuous families of local stable and unstable manifolds, and $\mu_\gamma\{\gamma \cap \Lambda\} > 0$ for every $\gamma \in \Gamma^u$.

A set A is an s-subset (resp. u-subset) of Λ if $\gamma \cap A \neq \emptyset$ implies $\gamma \subseteq A$ for any $\gamma \in \Gamma^{s(u)}$.

(P2) $R : \Lambda \to \mathbb{Z}^+$ is a *return time function* to Λ. Modulo a set of μ-measure zero, Λ is the disjoint union of s-subsets $\Lambda_j, j = 1, 2, \cdots$, with the property that for each j, $R|_{\Lambda_j} = R_j \in \mathbb{Z}^+$ and $f^{R_j}(\Lambda_j)$ is a u-subset of Λ. Moreover, for each n, the number of j such that $R_j = n$ is finite.

We refer to elements of $\Gamma^{u(s)}$ by $\gamma^{u(s)}$ and $|\det Df^u|$ denotes the unstable Jacobian of f with respect to μ_{γ^u}. Denote by $\gamma^s(x)$ and $\gamma^u(x)$ the stable and unstable leaves through x, respectively. For $x, y \in \Lambda$, there exists a separation time $s_0(x, y)$, depending only on the unstable coordinate, and numbers $C_0 \geq 1$, $\alpha < 1$ independent of x, y, such that the following hold.

(P3) For $y \in \gamma^s(x)$, $d(f^n x, f^n y) \leq C_0\alpha^n d(x, y)$ for all $n \geq 0$.[3]

(P4) For $y \in \gamma^u(x)$ and $0 \leq k \leq n < s_0(x, y)$,

(a) $d(f^n x, f^n y) \leq C_0\alpha^{s_0(x,y)-n}$;

(b) $\log \Pi_{i=k}^n \frac{\det Df^u(f^i x)}{\det Df^u(f^i y)} \leq C_0\alpha^{s_0(x,y)-n}$.

(P5) (a) For $y \in \gamma^s(x)$, $\log \Pi_{i=n}^\infty \frac{\det Df^u(f^i x)}{\det Df^u(f^i y)} \leq C_0\alpha^n$ for all $n \geq 0$.

(b) For $\gamma, \gamma' \in \Gamma^u$, if $\Theta : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ is defined by $\Theta(x) = \gamma^s(x) \cap \gamma'$, then Θ is absolutely continuous and $\frac{d(\Theta_*^{-1}\mu_{\gamma'})}{d\mu_\gamma}(x) = \Pi_{i=0}^\infty \frac{\det Df^u(f^i x)}{\det Df^u(f^i \Theta x)}$.

[3] As an abstract requirement, (P3) is slightly stronger than the inequality $d(f^n x, f^n y) \leq C_0\alpha^n$ stated in [Y2]. In practice, however, the stronger version holds for all systems for which Young towers have been constructed to date.

The meanings of the last three conditions are as follows: Orbits that have not "separated" are related by local hyperbolic estimates; they also have comparable derivatives. Specifically, **(P3)** and **(P4)**(a) are (nonuniform) hyperbolic conditions on orbits starting from Λ. **(P4)**(b) and **(P5)** treat more refined properties such as distortion and absolute continuity of Γ^s, conditions that are known to hold for $C^{1+\varepsilon}$ hyperbolic systems.

We say the generalized horseshoe (Λ, R) has *exponential return times* if there exist $C > 0$ and $\theta > 0$ such that for all $\gamma \in \Gamma^u$, $\mu_\gamma\{R > n\} \leq C\theta^n$ for all $n \geq 0$.

The setting described above is that of [Y2]; it does not involve holes. In this setting, we now identify a set $H \subset M$ (to be regarded later as the hole) and introduce a few relevant terminologies. Let (Λ, R) be a generalized horseshoe for f with $\Lambda \subset (M \setminus H)$.

Recall that (Λ, R) *respects* H if it satisfies conditions **(H.1)** and **(H.2)** of Section 1.A. In particular, **(H.1)** says that for every i and every ℓ with $0 \leq \ell \leq R_i$, $f^\ell(\Lambda_i)$ either does not intersect H or is completely contained in H.

When constructing the horseshoe which respects the hole in the sense of **(H.1)**, we still keep track of orbits that pass through H, i.e. we construct a horseshoe for the closed dynamical system (f, M) including ∂H as part of the singularity set, but we do not allow any escape at this stage. We do this to facilitate comparison between the dynamics of the open system and those of the closed system.

We say the horseshoe (Λ, R) is *mixing* if g.c.d.$\{R\} = 1$. If (Λ, R) respects H, let R_H denote the restriction of R to those Λ_i which return to Λ before entering H. Then when we treat H as a hole, we say the *surviving dynamics* are *mixing* if in addition g.c.d.$\{R_H\} = 1$.

3.2 From generalized horseshoes to Young towers

It is shown in [Y2] that given a map $f : M \to M$ with a generalized horseshoe (Λ, R), one can associate a Markov extension $F : \Delta \to \Delta$ which focuses on the return dynamics to Λ (and suppresses details between returns) . We first recall some facts about this very general construction, taking the opportunity to introduce some notation.

Let

$$\Delta = \{(x, n) \in \Lambda \times \mathbb{N} : n < R(x)\},$$

and define $F : \Delta \to \Delta$ as follows: For $\ell < R(x) - 1$, we let $F(x, \ell) = (x, \ell + 1)$, and define $F(x, R(x) - 1) = (f^{R(x)}(x), 0)$. Equivalently, one can view Δ as the disjoint union $\cup_{\ell \geq 0} \Delta_\ell$ where Δ_ℓ, the ℓ^{th} level of the tower, is a copy of $\{x \in \Lambda : R(x) > \ell\}$. This is the representation we will use. There is a natural projection $\pi : \Delta \to M$ such that $\pi \circ F = f \circ \pi$. In general, π is not one-to-one, but for each $\ell \geq 0$, it maps Δ_ℓ bijectively onto $f^\ell(\Lambda \cap \{R \geq \ell\})$ if f is invertible. We say (F, Δ) is mixing if (Λ, R) is mixing.

In the construction of (Λ, R), one usually introduces an increasing sequence of partitions of Λ into s-subsets representing distinguishable itineraries in the first n steps. These partitions induce a partition $\{\Delta_{\ell,j}\}$ of Δ which is finite on each level ℓ and is a (countable) Markov partition for F. We define a separation time $s(x, y) \leq s_0(x, y)$ by $\inf\{n > 0 : F^n x, F^n y$ lie in different $\Delta_{\ell,j}\}$.

We borrow the following language from (Λ, R) for use on Δ: For each ℓ, j, recall that $\Gamma^s(\pi(\Delta_{\ell,j}))$ and $\Gamma^u(\pi(\Delta_{\ell,j}))$ are the stable and unstable families defining the hyperbolic product set $\pi(\Delta_{\ell,j})$. We will say $\tilde{\gamma} \subset \Delta_{\ell,j}$ is an *unstable leaf* of $\Delta_{\ell,j}$ if $\pi(\tilde{\gamma}) = \gamma \cap \pi(\Delta_{\ell,j})$

for some $\gamma \in \Gamma^u(\pi(\Delta_{\ell,j}))$, and use $\Gamma^u(\Delta_{\ell,j})$ to denote the set of all such $\tilde{\gamma}$. Let $\Gamma^u(\Delta) = \cup_{\ell,j}\Gamma^u(\Delta_{\ell,j})$ be the set of all unstable leaves of Δ. *Stable leaves* of $\Delta_{\ell,j}$ and the families $\Gamma^s(\Delta_{\ell,j})$ and $\Gamma^s(\Delta)$ are defined similarly.

The measures μ_γ, $\gamma \in \Gamma^u(\Delta_0)$ are extended to Δ_ℓ, $\ell > 0$, by defining $\mu_\gamma(A) = \mu_\gamma(F^{-\ell}A)$ for all measurable $A \subset \Delta_\ell$. Thus $J_{\mu_\gamma}F$, the Jacobian of F with respect to μ_γ, satisfies $J_{\mu_\gamma}F \equiv 1$ except at return times. Let $J^u\pi$ denote the Jacobian of π with respect to the measures μ_γ on Δ and M respectively. Then F enjoys properties **(P3)-(P5)** at return times due to the identity $J^u\pi(F)J_{\mu_\gamma}F = |\det Df^u(\pi)|J^u\pi$ and the fact that $J^u\pi \equiv 1$ on Δ_0.

3.2.1 A reference measure on Δ

In each $\Delta_{\ell,j}$ we choose a representative leaf $\hat{\gamma} \in \Gamma^u(\Delta_{\ell,j})$. For any $\gamma \in \Gamma^u(\Delta_{\ell,j})$, let $\Theta_{\gamma,\hat{\gamma}} : \gamma \to \hat{\gamma}$ denote the holonomy map along Γ^s-leaves, i.e. $\Theta_{\gamma,\hat{\gamma}}(x) = \gamma^s(x) \cap \hat{\gamma}$. It will be convenient to define a new reference measure along unstable leaves, m_γ, by $dm_\gamma = \phi d\mu_\gamma$ where $\phi(x) = \prod_{i=0}^\infty \frac{J_{\mu_\gamma}F(F^ix)}{J_{\mu_\gamma}F(F^i(\Theta_{\gamma,\hat{\gamma}}x))}$. Given $\gamma' \in \Gamma^u(\Delta_0)$, if $\gamma \in \Gamma^u(\Delta_{\ell,j})$ satisfies $F(\gamma \cap S) = \gamma'$ for some s-subset S, then for $x \in \gamma \cap S$, define $J_\gamma F(x) = \frac{d(m_{\gamma'} \circ F)}{dm_\gamma}$. Elsewhere on Δ, $J_\gamma F \equiv 1$. Similarly, one defines $J_\gamma F^R(x)$ whenever $F^R(\gamma \cap S) = \gamma'$ for some s-subset S. For convenience, we restate Lemma 1 from [Y2], which summarizes the important properties of m_γ.

Lemma 3.1. *[Y2] Let γ, $\gamma' \in \Gamma^u(\Delta_{\ell,j})$.*

(1) Let $\Theta_{\gamma,\gamma'} : \gamma \to \gamma'$ be the holonomy map along Γ^s-leaves as above. Then $\Theta_* m_\gamma = m_{\gamma'}$.

(2) $J_\gamma F(x) = J_{\gamma'}F(y)$, $\forall x \in \gamma$, $y \in \gamma^s(x) \cap \gamma'$.

(3) $\exists C_1 > 0$ such that $\forall x, y \in \gamma$ with $s_0(x,y) \geq R(x)$, $\left|\frac{J_\gamma F^R(x)}{J_\gamma F^R(y)} - 1\right| \leq C_1 \alpha^{s(F^Rx,F^Ry)/2}$.

Moreover by (P5)(a), $e^{-C_0} \leq \phi \leq e^{C_0}$.

On Δ_0, we choose a transverse measure m^s on $\Gamma^u(\Delta_0)$ normalized so that $m^s(\Gamma^u(\Delta_0)) = 1$. Using the fact that $F : \Delta_\ell \to \Delta_{\ell+1}$ is simply rigid translation, we extend m^s to each $\Gamma^u(\Delta_{\ell,j})$. We define m to be the measure with factor measure m^s and measures m_γ on unstable leaves. Notice that in any $\Delta_{\ell,j}$, Lemma 3.1(1) implies that $m_\gamma(S) = m(S)$ for any s-subset $S \subseteq \Delta_{\ell,j}$ and $\gamma \in \Gamma^u(\Delta_{\ell,j})$. This feature of m_γ implies that m is a product measure on each $\Delta_{\ell,j}$. When disintegrating m on a particular $\Delta_{\ell,j}$, we maintain the convention that m^s is normalized, but m_γ is not.

Remark 3.2. *At this point in the application of Young towers, it is usual to define the quotient tower $\overline{\Delta} = \Delta/\sim$ where $x \sim y$ if $y \in \gamma^s(x)$. The resulting quotient system $(\overline{F}, \overline{\Delta})$ is expanding, allowing one to bring to bear the usual analysis of the transfer operator for expanding systems. However, since this requires the extra step of lifting our results from $\overline{\Delta}$ to Δ before projecting down to M, we find it simpler to work with the hyperbolic transfer operator on Δ directly, which we do below. Our approach also yields stronger results regarding the Hölder continuity of the quasi-invariant measures.*

3.2.2 Transfer operator

We define a metric along stable leaves which makes the distance between unstable leaves uniform. Fix $x \in \Delta_0$ and let $y \in \gamma^s(x)$. Let $\Theta : \gamma^u(x) \to \gamma^u(y)$ be the sliding map along stable leaves as above. Define $d_s(x, y) := \sup_{z \in \gamma^u(x)} d(z, \Theta z)$. We extend this metric to Δ_ℓ, $\ell > 1$, by setting $d_s(F^\ell x, F^\ell y) = \alpha^\ell d_s(x, y)$ for all $\ell < R(x)$ and $y \in \gamma^s(x)$. By **(P3)**,

$$d_s(F^n x, F^n y) \leq C_0 \alpha^n d_s(x, y) \qquad \text{for all } n \geq 0 \text{ whenever } y \in \gamma^s(x). \tag{2}$$

The class of test functions we use are required to be smooth along stable leaves only. Let $\mathcal{F}_b$ denote the set of bounded measurable functions on Δ. For $\varphi \in \mathcal{F}_b$ and $0 < r \leq 1$, define

$$K_s^r(\varphi) = \sup_{\gamma^s \in \Gamma^s(\Delta)} K^r(\varphi|_{\gamma^s}) \quad \text{where} \quad K^r(\varphi|_{\gamma^s}) = \sup_{x,y \in \gamma^s} |\varphi(x) - \varphi(y)| \, d_s(x, y)^{-r}.$$

If A is an s-subset of Δ, we define $|\varphi|_{\mathcal{C}_s^r(A)} = \sup_{\gamma^s \subset A} |\varphi|_{\mathcal{C}^0(\gamma^s)} + K^r(\varphi|_{\gamma^s})$ and let $\mathcal{C}_s^r(A) = \{\varphi \in \mathcal{F}_b : |\varphi|_{\mathcal{C}_s^r(A)} < \infty\}$.

For $h \in (\mathcal{C}_s^r(\Delta))'$ an element of the dual of $\mathcal{C}_s^r(\Delta)$, the transfer operator $\mathcal{L} : (\mathcal{C}_s^r(\Delta))' \to (\mathcal{C}_s^r(\Delta))'$ is defined by

$$\mathcal{L}h(\varphi) = h(\varphi \circ F) \qquad \text{for each } \varphi \in \mathcal{C}_s^r(\Delta).$$

When h is a measure absolutely continuous with respect to the reference measure m, we shall call its $L^1(m)$ density h as well. Hence $h(\varphi) = \int_\Delta \varphi h \, dm$. With this convention, $L^1(m) \subset (\mathcal{C}_s^r(\Delta))'$ and one can restrict $\mathcal{L}$ to $L^1(m)$. In this case,

$$\mathcal{L}^n h(x) = \sum_{y \in F^{-n}x} h(y)(J_m F^n(y))^{-1}$$

for each $n \geq 0$ where $J_m F^n$ is the Jacobian of F^n with respect to m.

Along unstable leaves, we define the metric $d_u(x, y) = \beta_0^{s(x,y)}$ for $y \in \gamma^u(x)$ and some $\beta_0 < 1$ to be chosen later. Let $\mathrm{Lip}^u(\varphi|_\gamma)$ denote the Lipschitz constant of a function φ along $\gamma \in \Gamma^u$ with respect to $d_u(\cdot, \cdot)$ and define $\mathrm{Lip}^u(\varphi) = \sup_{\gamma \in \Gamma^u(\Delta)} \mathrm{Lip}^u(\varphi|_\gamma)$. We define $\mathrm{Lip}^u(\Delta) = \{\varphi \in \mathcal{F}_b : \mathrm{Lip}^u(\varphi) < \infty\}$.

3.3 Definition of norms

We recall norms constructed in [D2] on which the transfer operator $\mathcal{L}$ has a spectral gap. We will show here that this spectrum is robust under perturbations in the form of small holes in Δ. We assume throughout that (F, Δ) has exponential return times, i.e. there exist constants $C > 0$, $\theta < 1$ such that $m(\Delta_\ell) \leq C\theta^\ell$.

Let $\mathcal{P} = \{\Delta_{\ell,j}\}$ denote the Markov partition for F. For each $k \geq 0$, define $\mathcal{P}^k = \bigvee_{i=0}^k F^{-i}\mathcal{P}$ and let $\mathcal{P}_{\ell,j}^k = \mathcal{P}^k|\Delta_{\ell,j}$. The elements $E \in \mathcal{P}_{\ell,j}^k$ are k-cylinders which are s-subsets of $\Delta_{\ell,j}$. For $\psi \in L^1(m)$ and $E \in \mathcal{P}^k$, define

$$⨍_E \psi \, dm = \frac{1}{m(E)} \int_E \psi \, dm.$$

Now choose $0 < q < p \leq 1$ and fix $1 > \beta_0 > \max\{\theta, \sqrt{\alpha}\}$ where α is from **(P3)**. Next, choose $1 > \beta \geq \max\{\beta_0^{(p-q)/p}, \alpha^q\}$.

For $h \in \mathrm{Lip}^u(\Delta)$, define the *weak norm* of h by $|h|_w = \sup_{\ell,j,k} |h|_{w(\mathcal{P}^k_{\ell,j})}$ where

$$|h|_{w(\mathcal{P}^k_{\ell,j})} = \beta_0^\ell \sup_{E\in\mathcal{P}^k_{\ell,j}} \sup_{|\varphi|_{\mathcal{C}^p_s(E)}\leq 1} \fint_E h\,\varphi\,dm. \tag{3}$$

Define the *strong stable norm* of h by $\|h\|_s = \sup_{\ell,j,k} \|h\|_{s(\mathcal{P}^k_{\ell,j})}$ where

$$\|h\|_{s(\mathcal{P}^k_{\ell,j})} = \beta^\ell \sup_{E\in\mathcal{P}^k_{\ell,j}} \sup_{|\varphi|_{\mathcal{C}^q_s(E)}\leq 1} \fint_E h\,\varphi\,dm. \tag{4}$$

For $\varphi \in \mathcal{C}^p_s(\Delta)$, define φ_E on $E \in \mathcal{P}^k_{\ell,j}$ by $\varphi_E(x) = m(E)^{-1}\int_{\gamma^u(x)\cap E} \varphi\,dm_\gamma$, for $x \in E$. Let $\tilde\varphi_E(x) = \varphi_E(\gamma^u(x))$ for $x \in \Delta_{\ell,j}$ be the extension of φ_E to $\Delta_{\ell,j}$. Note that $\tilde\varphi_E$ is well-defined since φ_E is constant on unstable leaves. In what follows, let $E_k \in \mathcal{P}^k_{\ell,j}$, $E_r \in \mathcal{P}^r_{\ell,j}$ for $r \geq k$.

We define the *strong unstable norm* of h by $\|h\|_u = \sup_{\ell,j,k} \|h\|_{u(\mathcal{P}^k_{\ell,j})}$ where

$$\|h\|_{u(\mathcal{P}^k_{\ell,j})} = \sup_{E_k\in\mathcal{P}^k_{\ell,j}} \sup_{E_r\subset E_k} \sup_{|\varphi|_{\mathcal{C}^p_s(E_r)}\leq 1} \beta^{\ell-k} \left| \fint_{E_r} h\,\varphi\,dm - \fint_{E_k} h\,\tilde\varphi_{E_r}\,dm \right|. \tag{5}$$

The *strong norm* of h is defined as $\|h\| = \|h\|_s + b\|h\|_u$, for some $b > 0$ to be chosen later.

We denote by $\mathcal{B}$ the completion of $\mathrm{Lip}^u(\Delta)$ in the $\|\cdot\|$-norm and by $\mathcal{B}_w$ the completion of $\mathrm{Lip}^u(\Delta)$ in the $|\cdot|_w$ norm.

3.4 Known spectral picture for $\mathcal{L} : \mathcal{B} \circlearrowleft$

The following proposition is [D2, Proposition 1.3]

Proposition 3.3. *[D2] Suppose (F,Δ) satisfies properties* **(P1)**-**(P5)** *and has exponential return times. Then there exists $\bar C > 0$ such that for each $h \in \mathcal{B}$ and $n \geq 0$,*

$$\begin{aligned} |\mathcal{L}^n h|_w &\leq \bar C |h|_w & (6)\\ \|\mathcal{L}^n h\|_s &\leq \bar C\beta^n\|h\|_s + \bar C|h|_w & (7)\\ \|\mathcal{L}^n h\|_u &\leq \bar C\beta^n\|h\|_u + \bar C\|h\|_s & (8)\end{aligned}$$

For any $1 > \tau > \beta$, there exists $N \geq 0$ such that $2\bar C\beta^N < \tau^N$. Choose $b = \beta^N$. Then,

$$\|\mathcal{L}^N h\| = \|\mathcal{L}^N h\|_s + b\|\mathcal{L}^N h\|_u \leq \bar C\beta^N(\|h\|_s + b\|h\|_u) + b\bar C\|h\|_s + \bar C|h|_w \leq \tau^N\|h\| + \bar C|h|_w.$$

The above represents the traditional Lasota-Yorke inequality. By [D2, Lemma 2.6], the unit ball of $\mathcal{B}$ is relatively compact in $\mathcal{B}_w$, so it follows from standard arguments that the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is bounded by β (see e.g. [B, HH]).

Theorem 3.4. *[D2, Theorems 1, 2] The operator $\mathcal{L} : \mathcal{B} \circlearrowleft$ is quasi-compact with essential spectral radius bounded by β. If F is mixing, then $\mathcal{L}$ has a spectral gap and there is a unique invariant probability measure $\tilde\nu_{SRB} \in \mathcal{B}$ with Lipschitz densities on $\gamma \in \Gamma^u(\Delta)$. In addition, $\tilde\nu_{SRB}$ projects to the SRB measure for f, i.e. $\pi_*\tilde\nu_{SRB} = \nu_{SRB}$.*

3.5 Proof of Theorem 1

Our strategy is to prove that the spectral gap for the transfer operator persists after the introduction of a small hole. We adopt the perturbative approach presented in [KL1].

Recall the definition of a Young tower respecting a hole H from Section 1.A (i.e. conditions **(H.1)** and **(H.2)**). Define $\mu^u(\Lambda) := \inf_{\gamma\in\Gamma^u}\mu_\gamma(\Lambda\cap\gamma)$. For a generalized horseshoe (Λ, R) respecting a hole H or a pair of holes H_1, H_2, we define

$$n(\Lambda, R; H) = \sup\{n \in \mathbb{Z}^+ : \text{ no point in } \Lambda \text{ falls into } H \text{ in the first } n \text{ iterates}\} \text{ and}$$
$$n(\Lambda, R; H_1, H_2) = \sup\{n \in \mathbb{Z}^+ : \text{no point in } \Lambda \text{ falls into } H_1 \triangle H_2 \text{ in the first } n \text{ iterates}\}$$

where $H_1\triangle H_2$ represents the symmetric difference between H_1 and H_2.

Let $\{H_t\}_{t\in I}$ be a well-parameterized sequence of holes as defined in Section 1.A. For $\varepsilon > 0$, we define $\mathbb{H}_\varepsilon = \{H_t : t \le \varepsilon\}$. The following uniform constants property is assumed for the tower construction.

(U) Uniform Constants. *Let $\{H_t\}_{t\in I}$ be as above. There exist constants $C_2, \kappa > 0$ and $\theta \in (0, 1)$ such that for all small enough $\varepsilon > 0$, we have the following:*

- *(a) For each pair $\sigma = (H_1, H_2) \in \mathbb{H}_\varepsilon \times \mathbb{H}_\varepsilon$,*
  - *(i) f admits a generalized horseshoe $(\Lambda^{(\sigma)}, R^{(\sigma)})$ respecting both H_1 and H_2, i.e., the combined boundaries $\partial H_1 \cup \partial H_2$;*
  - *(ii) $(\Lambda^{(\sigma)}, R^{(\sigma)})$ is mixing.*
- *(b) Each generalized horseshoe $(\Lambda^{(\sigma)}, R^{(\sigma)})$ from above can be constructed to have the following uniform properties:*
  - *(i) $\Lambda^{(\sigma_1)} \approx \Lambda^{(\sigma_2)}$,[4] for all $\sigma_1, \sigma_2 \in \mathbb{H}_\varepsilon \times \mathbb{H}_\varepsilon$*
  - *(ii) $\mu^u(\Lambda^{(\sigma)}) \ge \kappa$ and $\mu_\gamma\{R^{(\sigma)} > n\} < C_2\theta^n$ for all $n \ge 0$;*
  - *(iii)* **(P3)–(P5)** *hold with the constants C_0 and α.*

Remark 3.5. *The uniformity condition* **(U)** *has been proved for the billiard map associated with the periodic Lorentz Gas with small holes (see [DWY1, Proposition 2.2 and Section 3.1]). For the purposes of verifying the uniformity condition, an essential feature of the sequences of holes considered there is that the boundaries of the holes are transverse to the stable and unstable manifolds. Although it may at first seem a strong requirement, checking the items listed in* **(U)** *requires only a small modification of the tower construction necessary to build a tower for a single hole.*

For the remainder of the proof, we fix $\varepsilon > 0$ and assume that $\mathbb{H}_\varepsilon$ satisfies the uniformity conditions **(U)**. Given $H_1, H_2 \in \mathbb{H}_\varepsilon$, **(U)**(a)(i) immediately yields a tower (F, Δ) in which $\tilde{H}_1 = \pi^{-1}H_1$ and $\tilde{H}_2 = \pi^{-1}H_2$ are both countable unions of Markov partition elements $\Delta_{\ell,j}$. There are 3 towers we wish to compare: (F, Δ) which has cuts respecting both $\tilde{H}_1$ and $\tilde{H}_2$, but no holes have been removed; $(\mathring{F}_1, \Delta(\tilde{H}_1))$, the open system corresponding to the removal

[4] By $\Lambda^{(\sigma_1)} \approx \Lambda^{(\sigma_2)}$, we only wish to convey that both horseshoes are located in roughly the same region of the manifold M and not anything technical in the sense of convergence.

of $\tilde{H}_1$ from Δ; and $(\mathring{F}_2, \Delta(\tilde{H}_2))$, the open system corresponding to $\tilde{H}_2$. Note that $\mathring{F}_1$ and $\mathring{F}_2$ are both restrictions of the same map F.

We denote the transfer operators associated with these systems by $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Since $\Delta(\tilde{H}_1), \Delta(\tilde{H}_2) \subset \Delta$, we may consider all three operators acting on a single Banach space $(\mathcal{B}, \|\cdot\|)$ as defined in Section 3.3.

We let $\Delta^n(\tilde{H}_i)$ denote the set of points in Δ which have not escaped from the tower with hole $\tilde{H}_i$ by time n. Notice that for $h \in \mathrm{Lip}^u(\Delta)$,

$$\mathcal{L}_i h = 1_{\Delta \setminus \tilde{H}_i} \mathcal{L}(1_{\Delta \setminus \tilde{H}_i} h) = \mathcal{L}(1_{\Delta^1(\tilde{H}_i)} h) \text{ for } i = 1, 2. \tag{9}$$

Since $1_{\tilde{H}_i} \in \mathcal{B}$, it follows that $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the inequalities (6)-(8) from Proposition 3.3 with uniform constants $\bar{C} > 0$, $\beta < 1$.

In order to carry out the perturbation argument in [KL1], we introduce the following norm for operators $\mathcal{K} : \mathcal{B} \to \mathcal{B}_w$:

$$|||\mathcal{K}||| = \sup\{|\mathcal{K}h|_w : \|h\| \leq 1\}.$$

Lemma 3.6. *There exists $C > 0$ such that for any $H_1, H_2 \in \mathbb{H}_\varepsilon$,*

$$|||\mathcal{L} - \mathcal{L}_i||| \leq C(\beta^{-1}\beta_0)^{n(\Lambda,R;H_i)} \text{ for } i = 1, 2, \text{ and} \tag{10}$$

$$|||\mathcal{L}_1 - \mathcal{L}_2||| \leq C(\beta^{-1}\beta_0)^{n(\Lambda,R;H_1,H_2)}. \tag{11}$$

Proof. By the density of $\mathrm{Lip}^u(\Delta)$ in $\mathcal{B}$ and $\mathcal{B}_w$, it suffices to derive these inequalities for $h \in \mathrm{Lip}^u(\Delta)$. Now let $h \in \mathrm{Lip}^u(\Delta)$, $\|h\| \leq 1$. Fix $E \in \mathcal{P}^k_{\ell,j}$ and take $\varphi \in \mathcal{C}^p_s(E)$ with $|\varphi|_{\mathcal{C}^p_s(E)} \leq 1$.

We first consider the case when $\ell = 0$ and $E \in \mathcal{P}^k_{0,j}$. Note that $F^{-1}E$ is comprised of a countable union of $(k+1)$-cylinders, $F^{-1}E = \cup E'$, $E' \in \mathcal{P}^{k+1}_{\ell',j'}$. On level $\ell' \leq n(\Lambda, R; \tilde{H}_1)$, we have $1_{\Delta^1(\tilde{H}_1)} = 1$. Also, since $\Delta^1(\tilde{H}_1)$ is a union of 1-cylinders, we have either $1_{\Delta^1(\tilde{H}_1)}|_{E'} \equiv 0$ or $1_{\Delta^1(\tilde{H}_1)}|_{E'} \equiv 1$ for each $E' \subset F^{-1}E$ when $\ell' > n(\Lambda, R; \tilde{H}_1)$. Thus by (9),

$$\int_E (\mathcal{L} - \mathcal{L}_1) h\, \varphi\, dm = \sum_{E'} \int_{E'} (1 - 1_{\Delta^1(\tilde{H}_1)}) h\, \varphi \circ T\, dm \leq \sum_{\ell' \geq n(\Lambda,R;H_1)} \beta^{-\ell'} m(E') \|h\|_s |\varphi \circ F^n|_{\mathcal{C}^q_s(E')} \tag{12}$$

where $E' \subseteq \Delta_{\ell',j'}$. To estimate $|\varphi \circ F^n|_{\mathcal{C}^q_s(E')}$, take $x, y \in \gamma^s \subset E'$ and write

$$|\varphi \circ F^n(x) - \varphi \circ F^n(y)| \leq K^q_s(\varphi) d_s(F^n x, F^n y)^q \leq K^q_s(\varphi) C_0 \alpha^{qn} d_s(x,y)^q$$

by (2) since $q < p$. This together with $|\varphi \circ F^n|_\infty = |\varphi|_\infty$ implies $|\varphi \circ F^n|_{\mathcal{C}^q_s(E')} \leq C_0 |\varphi|_{\mathcal{C}^q_s(E)}$.

Due to bounded distortion given by Property **(P4)**(b) and Lemma 3.1, we have $\frac{m(E')}{m(E)} \leq C_1 \frac{m(E'_1)}{m(\Delta_0)}$ where E'_1 is the 1-cylinder containing E'. Thus by **(U)**(b)(ii), there exists $C > 0$ such that,

$$m(E') \leq C m(E'_1) m(E). \tag{13}$$

Now (12) becomes,

$$\int_E (\mathcal{L} - \mathcal{L}_1) h\, \varphi\, dm \leq C C_0 \sum_{\ell' \geq n(\Lambda,R;H_1)} \beta^{-\ell'} m(E'_1) m(E) \|h\| \leq C'(\beta^{-1}\theta)^{n(\Lambda,R;H_1)} m(E) \|h\|$$

since $\theta < \beta$. Dividing by $m(E)$ and taking the supremum over $\varphi \in \mathcal{C}_s^p(E)$ and $E \in \mathcal{P}_{0,j}^k$, we have

$$|(\mathcal{L} - \mathcal{L}_1)h|_{w(\mathcal{P}_{0,j}^k)} \le C'(\beta^{-1}\theta)^{n(\Lambda,R;H_1)}\|h\|. \tag{14}$$

Next consider $1 \le \ell \le n(\Lambda, R; H_1)$. Then (9) implies that $(\mathcal{L}h)|_{\Delta_\ell} = (\mathcal{L}_1 h)|_{\Delta_\ell}$, so

$$\int_E (\mathcal{L} - \mathcal{L}_1)h\,\varphi\,dm = 0.$$

Finally, we consider the case $\ell > n(\Lambda, R; H_1)$, then

$$\int_E (\mathcal{L} - \mathcal{L}_1)h\,\varphi\,dm = \int_{F^{-1}E} (1 - 1_{\Delta^1(\tilde{H}_1)})h\,\varphi \circ T\,dm \le \|h\|_s \beta^{-\ell+1} m(E)|\varphi|_{\mathcal{C}_s^q(E)},$$

since $m(F^{-1}E) = m(E)$. Taking the appropriate suprema in the definition of the weak norm, we obtain for $\ell \ge 1$,

$$|(\mathcal{L} - \mathcal{L}_1)h|_{w(\mathcal{P}_{\ell,j}^k)} \le (\beta^{-1}\beta_0)^{n(\Lambda,R;H_1)}\|h\|. \tag{15}$$

Combining (14) and (15) proves (10) for $i = 1$ since $\theta < \beta_0 < \beta$. The proof for $i = 2$ is identical.

To prove (11), replace $\mathcal{L}$ by $\mathcal{L}_2$ and the analogous estimates follow using the fact that $1_{\Delta^1(\tilde{H}_1)} = 1_{\Delta^1(\tilde{H}_2)}$ on all levels $\ell \le n(\Lambda, R; H_1, H_2)$. □

Let $H_1 = H_{t_1}$, $H_2 = H_{t_2} \in \mathbb{H}_\varepsilon$ (we allow the possibility that one of the holes is the infinitesimal hole H_0).

By definition, $\mathrm{dist}(H_1, H_2) \le |t_1 - t_2|$. By condition **(H.2)**, $d(f^\ell \Lambda, \mathcal{S}_{H_1} \cup \mathcal{S}_{H_2}) \ge \delta \xi_1^{-\ell}$ so that $n(\Lambda, R; H_1, H_2) \ge -\log(|t_1 - t_2|/\delta)/\log \xi_1$ for $|t_1 - t_2| < \delta$. Lemma 3.6 implies that

$$|||\mathcal{L} - \mathcal{L}_i||| \le C\delta^{-1}|t_i|^{\log(\beta_0^{-1}\beta)/\log \xi_1} \text{ for } i = 1, 2, \text{ and}$$
$$|||\mathcal{L}_1 - \mathcal{L}_2||| \le C\delta^{-1}|t_1 - t_2|^{\log(\beta_0^{-1}\beta)/\log \xi_1}.$$

Now the results of [KL1] imply that both the spectra and spectral projectors outside any disk of radius greater than β vary Hölder continuously in the size of the perturbation. Since the original dynamics are mixing by **(U)**(a)(ii), $\mathcal{L}$ has a spectral gap by Theorem 3.4. Thus there exists $\delta > 0$ such that the spectral gap for $\mathcal{L}$ is preserved for $\mathcal{L}_i$ for $t_i \le \delta$.

Let $\tilde{\mu}_i$ denote the physical quasi-invariant measure in $\mathcal{B}$ corresponding to the leading eigenvalue $\mathfrak{r}_i$ of $\mathcal{L}_i$, $i = 1, 2$. Then [DWY1, Theorem 4.4] implies that the escape rate from $\Delta(\tilde{H}_i)$ with respect to $\tilde{\mu}_i$, $-\rho(\tilde{\mu}_i)$, exists and equals $-\log \mathfrak{r}_i$. By [DWY1, Theorem 2], the escape rate $-\rho_i(\nu_{\mathrm{SRB}})$ from $M \backslash H_i$ with respect to ν_{SRB} equals $-\log \mathfrak{r}_i$ as well.

In particular, we have $|\mathfrak{r}_1 - \mathfrak{r}_2| \le C'|t_1 - t_2|^{\bar{\alpha}}$ for any $\bar{\alpha} < \log(\beta_0^{-1}\beta)/\log \xi_1$. This implies that $t \mapsto \rho(t)$ is a Hölder continuous function for $t \le \delta$. In addition, [KL1] also implies that the spectral projectors vary Hölder continuously so that the quasi-invariant measures $\tilde{\mu}_t$ vary Hölder continuously in the weak norm $|\cdot|_w$ for $t \le \delta$. Projecting these quasi-invariant measures to M, we obtain measures $\mu_t = \pi_* \tilde{\mu}_t$, which are quasi-invariant with respect to $f|_{M \backslash H_t}$. It follows from [D2, Lemmas 2.2 and 4.1] that the weak norm dominates the integral, i.e.,

$$|\tilde{\mu}_t(\varphi)| \le C|\tilde{\mu}_t|_w|\varphi|_{L^\infty}$$

for all $\varphi \in \mathcal{F}_b$ which are continuous on each element $\Delta_{\ell,j}$. Since a bounded continuous function φ on M lifts to a bounded function $\varphi \circ \pi$ on Δ that is continuous on each $\Delta_{\ell,j}$, the projected measures μ_t also vary Hölder continuously with t: Given $\varphi \in C^0(M)$, we have $|\mu_{t_1}(\varphi) - \mu_{t_2}(\varphi)| \le C''|t_1 - t_2|^{\bar{\alpha}}|\varphi|_{C^0(M)}$ for $t_1, t_2 \le \delta$.

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